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Comment on "Stronger subadditivity of entropy" by Lieb and Seiringer *Phys. Rev. A* **71**, 062329 (2005)

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Abstract

We show how recent results of Lieb and Seiringer can be obtained from repeated use of the monotonicity of relative entropy under partial traces, and explain how to use their approach to obtain tighter bounds in many situations.

In [7], Lieb and Seiringer (LS) proved an inequality which they view as stronger than the well-known strong subadditivity (SSA) of quantum entropy in a form equivalent to the contraction of relative entropy under partial traces. At first glance, this may seem inconsistent with recent work of Ibinison, Linden and Winter [1], who prove that this contraction is the *only* inequality satisfied by relative entropy. There is no real contradiction because, as LS acknowledge in [7], their results can be derived *from* SSA in the form of the monotonicity of relative entropy under completely positive trace preserving (CPT) maps [8, 10, 15]. Nevertheless, it seems worth restating the results in [7] in a way that makes clearer the connection with montonicity of relative entropy.

We need some notation. A density matrix is a positive semi-definite matrix ρ satisfying $\operatorname{Tr}\rho = 1$. The entropy of a density matrix ρ is given by $S(\rho) = -\operatorname{Tr}\rho\log\rho$, and the relative entropy of a pair of density matrices ρ, γ with $\ker(\gamma) \subset \ker(\rho)$ is given by

$$H(\rho,\gamma) = \operatorname{Tr}\rho\,\big(\log\rho - \log\gamma\big). \tag{1}$$

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SSA can be written as an inequality for the conditional entropy in the form

$$S(\rho_{BC}) - S(\rho_B) \geq S(\rho_{ABC}) - S(\rho_{AB}) \tag{2}$$

where ρ_{ABC} is a density matrix on the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, and the reduced density matrices are given by $\rho_{AB} = \mathrm{Tr}_C\, \rho_{ABC}$, $\rho_B = \mathrm{Tr}_A\, \rho_{AB} = \mathrm{Tr}_{AC}\, \rho_{ABC}$, etc. Since the conditional entropy $S(\rho_{AC}) - S(\rho_A)$ satisifies

$$S(\rho_{AC}) - S(\rho_A) = -H(\rho_{AC}, \rho_A \otimes \tfrac{1}{d_C} I_C) + \log d_C \tag{3}$$

with $\log d_C$ the dimension of $\mathcal{H}_C$, (2) can be rewritten as

$$H(\rho_{BC}, \rho_B \otimes \tfrac{1}{d_C} I_C) \leq H(\rho_{ABC}, \rho_{AB} \otimes \tfrac{1}{d_C} I_C). \tag{4}$$

The term $\frac{1}{d_C} I_C$ plays no role, except to ensure that the second argument of $H(\cdot,\cdot)$ is a density matrix.

Given a set of operators $\{K_m\}$ satisfying $\sum_m K_m^\dagger K_m = I$, Lieb and Seiringer [7] obtained an entropy inequality which can be stated in terms of a CPT map we call Λ_{LS} and write as

$$\Lambda_{\mathrm{LS}}(\rho) = \sum_m K_m \rho K_m^\dagger \otimes |m\rangle\langle m| \tag{5}$$

Thus, $\Lambda_{\mathrm{LS}}(\rho)$ is a block diagonal matrix with diagonal blocks $K_m \rho K_m^\dagger$. The condition $\sum_m K_m^\dagger K_m = I$ implies that the map

$$\Phi(\rho) = \sum_m K_m \rho K_m^\dagger. \tag{6}$$

is also a CPT map. By a slight modification[1] of the standard Lindblad-Stinespring ancilla representation [3, 4, 8, 11, 14] of a CPT map, we can represent both Φ and Λ_{LS} as partial traces on the same extended space $\mathcal{H}_A \otimes \mathcal{H}_D \otimes \mathcal{H}_E$ (with $\mathcal{H}_A$ the original Hilbert space) by defining

$$\sigma_{ADE}(\rho) = \sum_{mn} K_m \rho K_n^\dagger \otimes |m\rangle\langle n| \otimes |m\rangle\langle n| = V \rho V^\dagger \tag{7}$$

where V is a block column vector with elements $K_m \otimes |m\rangle \otimes |m\rangle$. It is easy to check that $V^\dagger V = I$ so that V is a partial isometry. Then

$$\Lambda_{\mathrm{LS}}(\rho) = \mathrm{Tr}_E\, \sigma_{ADE}(\rho) \tag{8}$$

[1]For details, see the Appendix of [3] and note that the representation above is equivalent to the familiar one [4] using a unitary U_{ADE} and a pure state $|\phi_{DE}\rangle\langle\phi_{DE}|$ such that $\sigma_{ADE}(\rho) = U_{ADE}\left(\rho \otimes |\phi_{DE}\rangle\langle\phi_{DE}|\right) U_{ADE}^\dagger$.

and

$$\Phi(\rho) = \mathrm{Tr}_{DE}\, \sigma_{ADE}(\rho) = \mathrm{Tr}_D\, \Lambda_{\mathrm{LS}}(\rho). \tag{9}$$

We similarly define $\tau_{ADE}(\gamma) = V\gamma V^\dagger$. Since $V^\dagger V = I$, one finds $S(\sigma_{ADE}) = S(\rho)$ and $H[\sigma_{ADE}, \tau_{ADE}] = H(\rho, \gamma)$, where we now suppress argument in $\sigma_{ADE}(\rho)$ etc. Therefore,

$$H[\sigma_A, \tau_A] \;\leq\; H[\sigma_{AD}, \tau_{AD}] \;\leq\; H[\sigma_{ADE}, \tau_{ADE}] \tag{10}$$

is equivalent to

$$H[\Phi(\rho), \Phi(\gamma)] \;\leq\; H[\Lambda_{\mathrm{LS}}(\rho), \Lambda_{\mathrm{LS}}(\gamma)] \;\leq\; H(\rho, \gamma) \tag{11}$$

for any pair of density matrices ρ, γ.

When $\rho = \rho_{ABC}$ and $\gamma = \rho_{AB} \otimes \frac{1}{d_C} I_C$ the inequality (11) can be written as

$$\begin{aligned} S(\rho_{ABC}) - S(\rho_{AB}) &\leq S\big[(\Lambda_{\mathrm{LS}} \otimes I_C)(\rho_{ABC})\big] - S\big[\Lambda_{\mathrm{LS}}(\rho_{AB})\big] && \text{(12a)} \\ &\leq S\big[(\Phi_{AB} \otimes I_C)(\rho_{ABC})\big] - S\big[(\Phi_{AB})(\rho_{AB})\big]. && \text{(12b)} \end{aligned}$$

The first inequality (12a) is the main theorem in [7].

When K_m acts nontrivially only on $\mathcal{H}_B$, LS obtained the following inequality, which is (9) in [7].

$$\begin{aligned} S(\rho_{ABC}) - S(\rho_{AB}) &\leq S\big[(\Lambda_{\mathrm{LS}} \otimes I_C)(\rho_{ABC})\big] - S\big[\Lambda_{\mathrm{LS}}(\rho_{AB})\big] \\ &\leq S(\rho_{AC}) - S(\rho_A) \end{aligned} \tag{13}$$

and the claim of "stronger" subadditivity rests on (13). In this situation, (8) becomes $\Lambda_{\mathrm{LS}}(\rho_{ABC}) = \mathrm{Tr}_E\, \sigma_{ABCE}$ with $\sigma_{ABCE} = U_{BE}\left(\rho_{ABC} \otimes |\phi_E\rangle\langle\phi_E|\right) U_{BE}^\dagger$. Then, as before,

$$S(\rho_{ABC}) - S(\rho_{AB}) \;=\; \log d_C - \; H(\sigma_{ABCE}, \sigma_{ABE} \otimes \tfrac{1}{d} I_C) \tag{14}$$

and, since $\sigma_{AC} = \mathrm{Tr}_{BE}\, \sigma_{ABCE} = \mathrm{Tr}_{BE}\, \rho_{ABC} \otimes |\phi_E\rangle\langle\phi_E| = \rho_{AC}$,

$$S(\rho_{AC}) - S(\rho_A) \;=\; \log d_C - \; H(\sigma_{AC}, \sigma_A \otimes \tfrac{1}{d} I_C). \tag{15}$$

Thus, (13) is equivalent to

$$H(\sigma_{ABCE}, \sigma_{ABE}) \geq H(\sigma_{ABC}, \sigma_{AB}) \geq H(\sigma_{AC}, \sigma_A) \tag{16}$$

where we have suppressed $\otimes \frac{1}{d} I_C$ in the second argument. In view of (16), the inequality (13) seems best viewed as an application of successive uses of SSA.

It is worth noting that (13) holds if $\Lambda_{\rm LS}$ is replaced by *any* CPT map for $\mathcal{H}_B$; it need not have the special form (5). Indeed, when $\Lambda_{\rm LS}$ and Φ_B are related as in (5) and (6), one finds that (12b) implies

$$\begin{aligned} S(\rho_{ABC}) - S(\rho_{AB}) &\leq S\big[(\Lambda_{\rm LS} \otimes I_C)(\rho_{ABC})\big] - S\big[\Lambda_{\rm LS}(\rho_{AB})\big] \\ &\leq S\big[(I_A \otimes \Phi_B \otimes I_C)(\rho_{ABC})\big] - S\big[I_A \otimes \Phi_B(\rho_{AB})\big] \\ &\leq S(\rho_{AC}) - S(\rho_A). \end{aligned} \tag{17}$$

Whether $\Lambda_{\rm LS}$ or Φ_B yields a "stronger" bound depends on whether one is trying to find a lower bound for $S(\rho_{ABC}) - S(\rho_{AB})$ or an upper bound for $S(\rho_{AC}) - S(\rho_A)$.

This suggests another way in which the results in [7] might be used to tighten bounds in some situations. Given any CPT map Φ, one can use its representation (6) to construct another CPT map $\Lambda_{\rm LS}$ as in (5). These maps satisfy the inequalities (11) and (12). Although the second inequality in each pair need not be strict, one expects that to be the generic situation when Φ does not already have the form $\Lambda_{\rm LS}$. Since the representation (6) is not unique, one can find a family of such bounds.

In [5], Lieb considered several natural ways of extending SSA to more than three parties and showed each was either an easy consequence of SSA or false. In [12], Pippenger gave a formal criterion for deciding whether or not an entropy inequality is "new", and independent of SSA, in terms of a convex cone of entropy vectors. Subsequently, Linden and Winter [9] found a new entropy inequality in the case of four parties and evidence [2] for another. However, the results in [1] imply that none of these can give a new inequality for the relative entropy. Thus, any strengthening of SSA in the "sandwich" sense of LS [7] must be reducible to the form (16).

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